Acceleration of hydrodynamic vortices in open systems

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A new class of exact solutions of hydrodynamic equations for an incompressible fluid (gas) at the presence of a bulk sink and uprising vertical flows of matter is considered. The acceleration of the rotation velocity of classical non-stationary vortices is conditioned by the joint action of the convective and Coriolis hydrodynamic forces (accelerations), which appear due to the converging radial flows of the matter in the region of a bulk sink. It is shown that there exist velocity profiles that nullify viscous terms in the Navier-Stokes equations and represent a vortex structure with a "rigid-body" rotation of its core and converging radial flows.

The concept of non-stationary vortices in open systems is applied to description of origination of power atmospheric vortices (whirlwinds, tornados, and typhoons). In the classical hydrodynamics a favorable condition for the origination and existence of such vortices is the exact nullification of the terms, which describe kinematic viscosity of an incompressible fluid. Such flows have the minimal rate of energy dissipation that corresponds to the "minimum entropy production principle", and therefore may relatively easily appear in favorable natural conditions.

1. Introduction

Hydrodynamical vortices and vortex-like flows are extremely widespread in nature. This is evidently proven by constant occurrence of powerful atmospheric vortices — cyclones, hurricanes, typhoons, tornados (see Nalivkin (1969), Bengsson & Lighthill (1982)), sandstorms, oceanic vortical flows — the so-called "rings", twirled streams (whirlpools) on rivers, turbulent vortices in wake swirls of ships, etc. An example of a huge stationary vortical structure is the big "Red Spot" on the surface of Jupiter, which has been observed by astronomers for several hundred of years.

One may ask a natural question: why such hydrodynamic vortex structures can exist for a long time in liquids and gases despite the finite viscosity of the medium? The answer to this question is partly given by the Rankine model (see Kundu (1990)) for a vortex in an incompressible viscous fluid where its azimuthal velocity profile is set to be linear in the radius r of a "rigid-body" rotation $v_\varphi(r) = \omega r$ with the angular velocity ω inside a certain cylindrical region $r \leqslant R_0$ of a radius R_0, and have the "differential" rotation $v_\varphi(r) = \omega R_0^2/r$ in the external region $r > R_0$. In this case, the terms with a bulk viscosity in

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the Navier-Stokes equations for an incompressible liquid (see Landau & Lifshits (1987)) are precisely nullified in the cylindrical system of coordinates. At the first sight, this corresponds to the non-dissipative vortical rotation of a liquid with a constant value of $\omega = \text{const}$ and with a sharp break of a profile $v_\varphi(r)$ at the point $r = R_0$.

However, the radial dependence of the velocity $v_\varphi(r)$ of real hydrodynamical vortices cannot have any singular points (in particular, jumps or breaks that correspond to jumps of the first derivative) and also must be a continuous analytical function of the radius r. Therefore, in a neighborhood of the point $r = R_0$, the function $v_\varphi(r)$ should be smooth, and there is a finite viscosity in this area, which results in dissipation of kinetic energy of a vortex, its spreading in space and decaying in time. Stationary or accelerating vortices can exist only due to inflow of energy from the external environment. At this, from all possible hydrodynamical flows and vortical movements in an incompressible liquid, easily arise, longer exist and dominate over others only those ones, whose viscosity effects are almost absent and energy dissipation plays an insignificant role. Such hydrodynamical flows and vortices correspond to the "minimum of entropy production principle" and are frequently enough observed in the nature.

In the present paper we will consider several examples of origination of non-stationary vortical structures in open multicomponent systems (solutions, homogeneous and heterogeneous mixtures), in which due to chemical or phase transformations a bulk sink (convergence) of the matter exists when one or several components drop out from the general collective hydrodynamical motion of the system. In this case, it is supposed that the chemical compound of the substance, as well as its density $\rho = \text{const}$, are constant in time and almost homogeneous in space due to dynamical and chemical balance with the surrounding environment.

We will show, that the growth of the rotation velocity of classical non-stationary vortices is caused by the joint action of convective and Coriolis hydrodynamic forces (accelerations) that arise due to converging radial flows of the substance flowing to the area of a bulk sink. The law of the velocity growth in time can be exponential, or can correspond to a nonlinear "explosive" regime of instability, when velocities formally reach infinite values in a finite time interval (under the condition of unlimited inflow of the substance from the environment).

A limit of the growth of velocity and kinetic energy of a vortex in a normal liquid (gas) in real conditions is caused by several factors: the friction between the liquid (or gas) and fixed solid surfaces; the development of turbulent dissipative processes, in particular, on the border of the vortex core at $r = R_0$, where a tangential jump of the velocity arises, and a connected with it small-scale instability of surface waves; the decrease of inflow of a substance from the outside, etc. We shall also note that at approaching of hydrodynamical velocity to the speed of sound, the compressibility of a liquid must be taken into account, which in turn accounts for dissipation due to the volumetric viscosity. All these factors will be considered and estimated below, but before that a simple mechanism of origination of a vortex in an incompressible one-component liquid (gas) due to existence of parallel to the vortex axis hydrodynamic flows, whose velocity depends on the longitudinal coordinate z, will be considered. Such a mechanism may explain the formation of a funnel in water or a sand-spout in a desert due to accelerated falling of the water in the hole or rising of the hot air in the gravitation field of the Earth.

2. One-component liquid with accelerated flows: the mechanism of formation of a funnel and a windspout

One of the most interesting paradoxes in hydrodynamics is the so-called "funnel effect" (see Kundu (1990), Sedov (1997)). It is usually assumed, that this effect is caused by conservation laws of the angular momentum of an incompressible fluid (gas) inside a given contour, and is accompanied by the accelerated rotation of a vortex at concentration of the vorticity of a flow $\omega = \mathrm{rot}\,\mathbf{v}$ due to the narrowing of the channel. Another approach to the problem of a funnel formation lies in the assumption (see Goldshtik, Shtern & Yavorsky (1989)) about origination of the angular momentum at zero initial vorticity as a result of instability of cylindrically-symmetric flow in a liquid (the flooded jet) in relation to axially-asymmetric left- and right-spiral perturbations with ejection of rotation of a certain sign to the infinity at the expense of a flow (convective instability) and accumulation of a rotation motion of the other sign (absolute instability).

Let us show, that there exists one additional simple mechanism of a vortex formation in an incompressible liquid (gas), which is present in a gravitational field and includes vertical ascending or descending flows, whose velocities depend on the coordinate z (along the vertical axis of the vortex).

Consider the Navier-Stokes equations for an axially-symmetric motion of an incompressible viscous fluid (gas) in cylindrical coordinates (see Landau & Lifshits (1987)):

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right), \tag{2.1}$$

$$\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} = \nu \left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r^2} \right), \tag{2.2}$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + \nu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right), \tag{2.3}$$

where v_r, v_φ and v_z are radial, azimuthal and axial components of the hydrodynamical velocity $\mathbf{v}$, P and ρ are the pressure and the density of a liquid (gas), $\nu = \eta/\rho$ is the coefficient of kinematic viscosity, and g is the acceleration due to gravity, which is directed to the opposite direction of the axis z.

Equations (2.1)–(2.3) are completed with the continuity equation

$$\mathrm{div}\,\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0. \tag{2.4}$$

Let us notice, that in equations (2.1) and (2.2), dependencies of v_r and v_φ on z are not taken into account only for simplification.

In the case of a plane vortical rotation, when $v_r = v_z = 0$, equations (2.1) and (2.2) become:

$$\frac{v_\varphi^2}{r} = \frac{1}{\rho}\frac{dP}{dr}, \tag{2.5}$$

$$\frac{\partial v_\varphi}{\partial t} = \nu \left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r^2} \right). \tag{2.6}$$

If the radial dependence of the azimuthal velocity $v_\varphi(r)$, according to the Rankine model of a vortex (see Kundu (1990)), is chosen to be

$$v_\varphi(r) = \begin{cases} \omega r, & r \leqslant R_0, \\ \omega R_0^2/r, & r > R_0, \end{cases} \tag{2.7}$$

then the right hand side of equation (2.6) is identically equal to zero, which corresponds to the rotation of an incompressible liquid (gas) with a constant angular velocity $\omega = \text{const}$.

At this, the distribution of hydrodynamical pressure $P(r)$, according to (2.1), is in the so-called 'cyclostrophic' rotation regime and has the form:

$$P(r) = \begin{cases} P_0 + \rho\omega^2 r^2/2, & r \leqslant R_0, \\ P_\infty - \rho\omega^2 R_0^4/2r^2, & r > R_0, \end{cases} \tag{2.8}$$

where $P_0 = P_\infty - \rho\omega^2 R_0^2$ is the pressure on the vortex axis, and P_∞ is the pressure at large distances (i.e. at $r \to \infty$).

In the Rankine model, the radius of the vortex core R_0 is not determined, but if in a liquid (gas) there exists a cylindrically-symmetric flow (the flooded jet) with a velocity v_z, then its radius will determine the size R_0 in equation (2.7). Let us assume, that the flow velocity depends linearly on the coordinate z and does not depend on r in the region $r \leqslant R_0$, i.e. we have the profile:

$$v_z(z) = \begin{cases} v_{z0} + \alpha z, & r \leqslant R_0, \\ 0, & r > R_0. \end{cases} \tag{2.9}$$

In this case the continuity equation (2.4) is satisfied for the following radial dependence of velocity v_r, which is continuous at $r = R_0$:

$$v_r(r) = \begin{cases} -\dfrac{1}{2}\alpha r, & r \leqslant R_0, \\ -\dfrac{1}{2}\alpha \dfrac{R_0^2}{r}, & r > R_0. \end{cases} \tag{2.10}$$

Let us notice, that the above mentioned structure of the velocities (2.7), (2.9) and (2.10) identically nullify the viscous terms in equations (2.1)–(2.3). At the same time, the diagonal components of the viscous stress tensor are different from zero for these profiles, which results in the following expression for the rate of changing of the kinetic energy of a vortex due to dissipation (per unit of length of a vortex along the axis z):

$$\left(\frac{dE_{\text{kin}}}{dt}\right)_{\text{dis}} = -4\pi\rho\nu\left[\frac{3}{2}\alpha^2 + \omega^2(0)\right]R_0^2. \tag{2.11}$$

Substituting expressions (2.7) and (2.10) in equation (2.2), we get

$$\frac{d\omega}{dt} = \begin{cases} \alpha\omega, & r \leqslant R_0, \\ 0, & r > R_0. \end{cases} \tag{2.12}$$

Equation (2.12) is a result of that fact that in the inner region ($r \leqslant R_0$) the convertive and Coriolis forces at $v_r \neq 0$ are added up, while in the external region ($r > R_0$) they mutually compensate each other.

If the parameter α is constant in time ($\alpha = \text{const}$, $\dfrac{d\alpha}{dt} = 0$) and positive $\alpha > 0$, then from equation (2.12) it follows that inside the region $r \leqslant R_0$ the angular velocity of a vortex grows in time according to the exponential law

$$\omega(t) = \omega(0)\, e^{\alpha t}, \tag{2.13}$$

provided that a nonzero initial vorticity $\omega(0) \neq 0$ exists in the liquid (gas), whereas in the external region $\omega = \omega(0) = \text{const}$. Thus, on the border of a vortex core $r = R_0$, a jump of the azimuthal velocity that exponentially grows in time arises (see section 4.2).

Let us emphasize, that the energy dissipation (2.11) of the non-stationary vortical motion does not depend on time t, and is determined only by the initial vorticity $2\omega(0)$.

Thus, the dissipation remains small, despite the fast increase of the angular velocity of a "rigid-body" rotation of the vortex core. This means that non-stationary vortices are not suppressed by the dissipation at least at an initial stage of their developments.

On the other hand, substituting expressions (2.7), (2.9) and (2.10) in equations (2.1) and (2.3), we get the following equations for the determination of the hydrodynamical pressure:

$$\frac{\partial P}{\partial r} = \begin{cases} \rho r \left[\omega^2(t) - \dfrac{\alpha^2(t)}{4} + \dfrac{1}{2}\dfrac{d\alpha}{dt}\right], & r \leqslant R_0, \\ \dfrac{\rho R_0^4}{r^3}\left[\omega^2(0) + \dfrac{\alpha^2(t)}{4} + \dfrac{1}{2}\dfrac{d\alpha}{dt}\dfrac{r^2}{R_0^2}\right], & r > R_0, \end{cases} \tag{2.14}$$

$$\frac{\partial P}{\partial z} = \begin{cases} -\rho\left[g + \alpha v_{z0} + z\left(\dfrac{d\alpha}{dt} + \alpha^2\right)\right], & r \leqslant R_0, \\ -\rho g, & r > R_0. \end{cases} \tag{2.15}$$

Let us notice, that equations (2.14) and (2.15) indicate on the existence and increasing in time of a jump of the first derivatives of the pressure $\partial P/\partial r$ and $\partial P/\partial z$ on the surface of the vortex core $r = R_0$. From equation (2.15), a possibility of existence of a non-stationary solution with $\frac{d\alpha}{dt} \neq 0$ follows, namely:

$$\frac{d\alpha}{dt} + \alpha^2(t) = 0, \quad \frac{\partial P}{\partial z} \pm \rho g = 0. \tag{2.16}$$

The first equation (2.16) at $\alpha > 0$ has the following solution:

$$\alpha(t) = \frac{\alpha_0}{1 + \alpha_0 t}, \quad \alpha_0 \equiv \alpha(0) > 0, \tag{2.17}$$

whereas the second equation of (2.16) corresponds to the hydrostatic pressure distribution in the whole space, and the sign $(+)$ corresponds to an ascending flow (the axis z is directed upwards), while the sign $(-)$ corresponds to a descending flow (the axis z is directed downwards). In this case, equation (2.12) in the region $r \leqslant R_0$ has a solution

$$\omega(t) = \omega(0)\exp\left\{\int_0^t \frac{\alpha_0\, dt'}{1 + \alpha_0 t'}\right\} = \omega(0)(1 + \alpha_0 t), \tag{2.18}$$

which corresponds to the linear growth of vortex rotation velocity in time.

At last, we notice that equation (2.12) under the condition of $\alpha(t) = \omega(t) > 0$ takes the form:

$$\frac{d\omega}{dt} - \omega^2(t) = 0. \tag{2.19}$$

The solution of the nonlinear equation (2.19) corresponds to the so-called "explosive" instability:

$$\omega(t) \equiv \alpha(t) = \frac{\omega(0)}{1 - \omega(0)t}, \tag{2.20}$$

when in a finite time interval $t_0 = 1/\omega(0) \equiv 1/\alpha(0)$ the angular velocity of a fluid rotation $\omega(t)$ and the derivative of the axial velocity with respect to z, i.e. $\alpha(t) \equiv \partial v_z/\partial z$, formally approach infinity, although they are actually limited from the above due to the effects of compressibility of a liquid.

The positive sign of α corresponds to the growth of velocity of the flow along the axis z directed by velocity v_{z0}. Thus, in a descending flow of a liquid which flows out through a hole at the bottom under the action of gravitation and accelerates along the axis z by the linear law (2.9), we get an exponential (2.13), linear (2.18), or "explosive" (2.20) laws of the acceleration of vortex rotation in time. At this, the amount of a liquid, which flows out, is completely compensated by the inflow of the same amount of the liquid with velocity of the converging radial flow (2.10) from the surrounding region, which is considered as a large enough reservoir of the substance.

Such a simple model explains the funnel formation in a bath at the opening of a hole or whirlpool formation on a river in the place of a sharp deepening of the bottom. The deceleration of the rate of rotation velocity growth of a liquid (water) in the stationary regime is caused by the friction with fixed solid surfaces as well as by the energy dissipation on the tangential jump of the azimuthal velocity v_φ at the point $r = R_0$ (see section 4.2).

This model may also explain the origination of sandy tornados in deserts. Due to a strong heating of some sites of a surface of a sandy ground by the sunlight (the darkest or located perpendicularly to the solar rays), the nearby air gets warm locally and starts rising upwards with acceleration under the action of the Archimedean force as less heavy. If this acceleration in quasi-stationary conditions corresponds approximately to the linear in z law (2.9), then at $\alpha > 0$ we again obtain the exponential law of rotation velocity growth of a vortex (2.13). At this, the accelerated decrease in time of the pressure near the vortex axis $r = 0$ with the increase of $\omega(t)$ and $\alpha(t)$ leads to the suction of the sand deep into the vortex and to formation of visible tornados, which are frequently observed in deserts.

3. Vortices in multi-component open systems with chemical and phase transformations

Consider a multi-component system in which chemical processes or phase transitions are taking place. Partial balance equations of substances for each component (phase) look like (see Sedov (1997)):

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \mathbf{v}_i) = Q_i, \quad \frac{dM_i}{dt} = \int_V Q_i \, dV, \tag{3.1}$$

where ρ_i and $\mathbf{v}_i$ are density and hydrodynamical velocity of the i-th component, M_i is the mass of this component in a volume V, Q_i is the capacity of a source ($Q_i > 0$) or sink ($Q_i < 0$) of this component due to chemical reactions or phase transformations.

For a closed system the conservation law of the total mass of all N components holds true:

$$M = \sum_{i=1}^{N} M_i = \text{const}, \quad \sum_{i=1}^{N} Q_i = 0, \tag{3.2}$$

and therefore the general continuity equation can be written as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \quad \rho = \sum_{i=1}^{N} \rho_i, \quad \rho \mathbf{v} = \sum_{i=1}^{N} \rho_i \mathbf{v}_i. \tag{3.3}$$

For open systems with a bulk sink or source of the substance and its unlimited inflow from the environment under conditions of dynamic and chemical balance, one or several components can appear in the system or drop out from the general collective motion of the matter (liquid or gas) as a result of chemical reactions or phase transformations. At the same time, diffusion and hydrodynamical flows support constant density and chemical potential in the open system, i.e. under stationary conditions it can be with a good approximation assumed that $\rho = \text{const}$. In this case the effective continuity equation can be written in the form

$$\operatorname{div}\mathbf{v} = Q/\rho \neq 0, \tag{3.4}$$

where Q is a certain capacity of a source ($Q > 0$) or sink ($Q < 0$) of the substance in a system or in a some part of its volume V.

Further we shall consider systems with a homogeneous in space bulk sink of matter in a some finite cylindrical region with the radius R_0. In cylindrical coordinates, equation (3.4) looks like (at the absence of longitudinal flows, $v_z = 0$):

$$\operatorname{div}\mathbf{v} \equiv \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = \begin{cases} -|Q|/\rho \equiv -1/\tau, & r \leqslant R_0, \\ 0, & r > R_0. \end{cases} \tag{3.5}$$

From the last expression it follows that the radial velocity can be set as

$$v_r(r) = \begin{cases} -\beta r, & r \leqslant R_0, \\ -\beta R_0^2/r, & r > R_0, \end{cases} \tag{3.6}$$

where $\beta = 1/2\tau > 0$. We notice, that $|Q|$, τ and β can be either constants, or variables in time, but must be homogeneous in space, i.e. the condition $\nabla \operatorname{div}\mathbf{v} = 0$, which is necessary and sufficient for the use of the Navier-Stokes equations (2.1) and (2.2) for an incompressible liquid, is satisfied.

Substituting expressions (2.7) and (3.6) for azimuthal and radial velocity profiles in equations (2.1) and (2.2), we get the equation for determination of hydrodynamic pressure (compare with (2.14)):

$$\frac{\partial P}{\partial r} = \begin{cases} \rho r \left(\omega^2 - \beta^2 + \dfrac{d\beta}{dt}\right), & r \leqslant R_0, \\ \dfrac{\rho R_0^4}{r^3}\left(\omega^2 + \beta^2 + \dfrac{d\beta}{dt}\dfrac{r^2}{R_0^2}\right), & r > R_0, \end{cases} \tag{3.7}$$

and also the equation for the angular velocity of rotation of the liquid (gas) in a vortex (compare with (2.12)):

$$\frac{d\omega}{dt} = \begin{cases} 2\beta\omega, & r \leqslant R_0, \\ 0, & r > R_0. \end{cases} \tag{3.8}$$

In the case of a constant sink $|Q| = \text{const}$ and $\beta = \text{const}$ we get from (3.8) the exponential law of acceleration of a vortical rotation (at $\omega(0) \neq 0$):

$$\omega(t) = \omega(0)\, e^{t/\tau}. \tag{3.9}$$

If the capacity of the surrounding reservoir is finite, and the velocity of inflow (and also the sink) of a substance in the dynamic equilibrium state decreases in time, for instance, by the exponential law

$$\beta(t) \equiv \frac{1}{2}\frac{|Q(t)|}{\rho} \equiv \beta_0\, e^{-t/t_0}, \quad \text{i.e.} \quad \tau(t) = \tau_0\, e^{t/t_0}, \tag{3.10}$$

then the time dependence of the angular velocity of a vortex rotation, according to (3.8), looks like

$$\omega(t) = \omega(0)\, \exp\left\{\frac{t_0}{\tau_0}\left(1 - e^{-t/t_0}\right)\right\} \tag{3.11}$$

and tends to the maximum value $\omega_\infty = \omega(0)\, e^{t_0/\tau_0}$ at $t \to \infty$, whereas during the time $t \ll t_0$ the exponential growth of velocity $\omega(t) \approx \omega(0)\, e^{t/\tau_0}$ remains.

From equation (3.8) it follows, that at $\beta > 0$ and either sign of ω, there exists another non-stationary solution at $\beta = \frac{1}{2}|\omega(t)|$, when equation (3.8) becomes (for $r \leqslant R_0$):

$$\frac{d|\omega|}{dt} - \omega^2(t) = 0. \tag{3.12}$$

The solution to this equation corresponds to the nonlinear instability of the "explosive" type (compare with (2.19)):

$$|\omega(t)| = \frac{|\omega(0)|}{1 - |\omega(0)|t}. \tag{3.13}$$

For a finite interval of time $t_0 = 1/|\omega(0)|$, the angular velocity $\omega(t)$ as well as $\beta(t)$ formally reach their infinite values.

Thus, the presence of a bulk sink together with the unlimited inflow of a substance in open systems with chemical reactions or phase transformations, due to nonlinear hydrodynamical forces, which arise under the action of converging flows $v_r < 0$, results in the acceleration of vortices even at the absence of axial flows ($v_z = 0$). An example of such a natural mechanism of vortex acceleration due to chemical reactions can be origination of the so-called "fiery tornados", which arise during big fires in closed volumes at the formation, on the one hand, of solid combustion materials, in particular solid compounds of oxygen (oxides) that completely exclude oxygen from the air, and, on the other hand, at the presence of free inflow (draught), which creates radial and ascending flows of the external air enriched with oxygen.

Moreover, during the dissolution of solid crystals in a liquid (for example, manganic-sour potassium in water), when the heavier solution is immersed downwards by gravity, and a new pure solvent with origination of converging radial flows goes to its place, a hydrodynamical vortex may arise, which due to friction will force the dissolving crystals to rotate.

4. Origination of tornados and typhoons during formation of dense cloud systems

Let us consider one more concrete example of origination of vortices in an open non-equilibrium heterogeneous system with phase transformations, namely, in a humid atmosphere during water vapor condensation in rain clouds where tornados and typhoons might originate.

Despite the colossal quantity of available observable data on tornados and typhoons in natural conditions and numerous attempts of modeling these powerful atmospheric vortices in laboratory plants and computer simulations (see Bengsson & Lighthill (1982)), the true reasons for origination and development of such phenomena are not yet understood completely. The majority of theoretical models of tornados and typhoons can be reduced to the well-known "funnel effect" in hydrodynamics (see Kundu (1990), Sedov (1997)). At this, it is supposed that the radial squeeze of twirled descending flows of dense humid air is being realized by the air masses, which are flowing in the region of low atmospheric pressure.

In Ref. (Pashitskii (2002)) an essentially new mechanism of tornado and typhoon origination during formation of dense cloud systems was introduced. This the mechanism is directly connected to the process of intensive condensation of water vapor at cooling of the humid air below the dew-point. A "vapor – liquid" phase transition with the formation of weighed water droplets in a cloud (fog) is accompanied by a large decrease (about 800 times) of the specific volume which is occupied by water molecules. This means that the condensation of vapor should result in essential (almost twice at initial

100% humidity of the air) decrease of the density of a gaseous component of the two-phase heterogeneous system "air – water drops" in comparison with the initial density of a chemically homogeneous gas mixture "air – water vapor". It is obvious that this is equivalent to the existence of a bulk sink (convergence) of a substance inside the cloud and origination of the concentration gradient of water molecules on the border of a cloud.

As it follows from the Boltzmann equation for a mixture of gases with close by weight molecules (see Lifshits & Pitaevskii (1981)), the diffusion flow of one of the components under the action of the corresponding concentration gradient, as a result of intermolecular collisions, creates an average hydrodynamical flow of the gas mixture as a whole. Since the cloud is a thermodynamically non-equilibrium open system with the unlimited inflow of a substance from the outside, the presence of a bulk sink and the humidity gradient caused by the condensation of moisture should result in appearance of converging hydrodynamical flows of the humid air from the surrounding areas of the atmosphere. Under conditions of a dynamical balance, such flows retain a constant in time and almost homogeneous in space average density $\rho = \text{const}$, which is equivalent to the condition of incompressibility of the medium.

As has been shown in the previous section, within the framework of the hydrodynamical model of an incompressible viscous liquid with a bulk sink (convergence) of the substance and converging radial flows, under the joint action of convective and Coriolis forces an instability of a vortical "rigid-body" rotation appears, which can evolve in time either by the exponential law (in the case of constant sink and inflow of substance), or by a scenario of the "explosive" type when the infinite rotation velocity is reached in a finite time interval (under the condition of simultaneous unlimited increase of velocities of the sink and the inflow).

We will show, that such an instability can be one of the reasons of origination and acceleration of powerful tornadoes and typhoons during the formation of dense cloud systems (see also Pashitskii (2002)).

4.1. *Balance of substance during the condensation of rain clouds*

The rain cloud represents an open non-equilibrium two-phase system (a mixture of the air and a fog) that self-organizes due to condensation of water vapor and keeps small water drops in a certain volume by converging flows of the air and the Stokes friction forces.

Consider a cloud of a cylindrical form of the radius R_0 and height h. As inside a forming cloud the process of vapor condensation from the humid air, cooled below the dew-point T_{dew} (at the given pressure distribution P) takes place with the certain velocity, the effective balance equation of the substance inside a cloud, with account of horizontal (radial) and vertical (ascending or descending) air flows from the surrounding atmosphere, and a bulk sink of the substance with capacity $Q < 0$, can be written as

$$2\pi R_0 h v_r(R_0) + \pi R_0^2 [v_z(h) - v_z(0)] = -\frac{|Q|}{\rho} \pi R_0^2 h. \tag{4.1}$$

Substituting expressions (2.9) and (2.10) in equation (4.1), we get the relation

$$2\beta = \alpha + \frac{|Q|}{\rho}, \tag{4.2}$$

which corresponds to the effective continuity equation (compare with (2.4) and (3.5))

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = -\frac{|Q|}{\rho} \equiv -\frac{1}{\tau}. \tag{4.3}$$

Thus, at $\alpha > 0$ there is a strengthening of the nonlinear instability, which results in acceleration of a vortex rotation in the region $r \leqslant R_0$ by the following law:

$$\omega(t) = \omega(0) \exp\left\{\int_0^t dt' \left[\alpha(t') + \frac{1}{\tau(t')}\right]\right\}. \tag{4.4}$$

At the same time, in the external region $r > R_0$, in agreement with (3.8), the angular velocity remains constant and is equal to the initial value $\omega(0)$. This means that on the border of the vortex core at $r = R_0$ a tangential discontinuity of azimuthal velocity arises and grows in time. This jump at $\alpha = \text{const}$ and $|Q| = \text{const}$ equals

$$\Delta v_\varphi(R_0, t) \equiv V_0(t) = \omega(0) R_0 [e^{2\beta t} - 1], \tag{4.5}$$

and in the case of an "explosive" instability we have

$$V_0(t) = \omega(0) R_0 \left[\frac{1}{1 - \omega(0)t} - 1\right]. \tag{4.6}$$

4.2. *Instability of the tangential jump of velocity on the border of the tornado core*

As is well-known (see Landau & Lifshits (1987)), a tangential jump of velocity in hydrodynamics results in origination of instability of surface perturbations of an incompressible fluid. The dispersion equation of small perturbations in linear approximation with account of a finite viscosity looks like (see Pashitskii (2002)):

$$(\omega - kV_0)(\omega - kV_0 + i\nu k^2) = -\omega(\omega + i\nu k^2), \tag{4.7}$$

where $k = 2\pi/\lambda$ is the wave number of surface waves with the length λ, which are distributed along a flat surface. The equation (4.7) with a good approximation can be also applied to the case of the cylindrical surface of a radius R_0 if we consider $kR_0 \gg 1$.

The frequency $\omega_0(k) \equiv \mathrm{Re}\,\omega(k)$ and the increment $\gamma(k) \equiv \mathrm{Im}\,\omega(k)$ of such short-wave surface perturbations on the tangential jump of velocities with transversal displacements of a surface along the z axis and the propagation along the x axis (at $k > 0$)

$$\zeta(x, z, t) = \zeta_0\, e^{-k|z|}\, e^{\gamma t} \exp(ikx - i\omega_0 t), \tag{4.8}$$

according to dispersion equation (4.7), are equal to

$$\omega_0(k) = \frac{kV_0}{2}, \quad \gamma(k) = \frac{1}{2}\left[\sqrt{k^2 V_0^2 + \nu^2 k^4} - \nu k^2\right]. \tag{4.9}$$

If $k \ll V_0/\nu$ (but $kR_0 \gg 1$), then the increment is $\gamma(k) = kV_0/2$. At $k \gg V_0/\nu$, according to (4.9), the increment tends to its maximum value (at $k \to \infty$):

$$\gamma_\infty = V_0^2/2\nu. \tag{4.10}$$

It is worth noting, that relations (4.7) – (4.10) were obtained under the assumption of a constant value of the velocity jump $V_0 = \text{const}$. But if the condition $\gamma_\infty \gg 2\beta$ is satisfied, i.e. if the jump of the velocity (4.5) grows in time very slowly (adiabatically) in comparison with development times of the unstable short-wave excitations with increment (4.10), it is possible to present approximately the time evolution of the amplitudes of surface perturbations at $z = 0$ in the form

$$|\zeta(t)| \simeq \zeta_0\, e^{\gamma_\infty t} = \zeta_0 \exp\left\{\frac{\omega^2(0) R_0^2 t}{2\nu} \left(e^{2\beta t} - 1\right)^2\right\} \tag{4.11}$$

for the exponential instability of a vortex (for $\beta = \text{const}$), or

$$|\zeta(t)| = \zeta_0 \,\exp\left\{\frac{\omega^2(0)R_0^2 t}{2\nu}\left[\frac{1}{1-\omega(0)t} - 1\right]^2\right\} \tag{4.12}$$

in the case of the "explosive" instability given by (2.19) and (3.13).

During the time interval $t \ll 1/2\beta$, when $V_0(t) \simeq 2\beta\omega(0)R_0 t$, we have

$$|\zeta(t)| \simeq \zeta_0 \,\exp\left\{\frac{2\beta^2\omega^2(0)R_0^2 t^3}{\nu}\right\}, \tag{4.13}$$

instead of the "exponent to the exponent" law (4.11).

A more detailed analysis of time evolution of small surface perturbations at a variable velocity $V_0(t)$ can be obtained in the case where $\nu = 0$ with the help of the following linear differential equation:

$$\frac{d^2\zeta}{dt^2} - kV_0(t)\frac{d\zeta}{dt} + \frac{1}{2}k^2V_0^2(t)\zeta(t) = 0. \tag{4.14}$$

If $V_0(t) \sim t$, then the dominate time dependence of the solution (4.14) looks like

$$\zeta(t) \sim \exp\left\{at^2 - bt\right\}, \tag{4.15}$$

i.e. it is slower, than the dependence (4.13). But if one takes into the account the fact, that at $\nu \to 0$ the inequality $k \ll V_0(t)/\nu$ always takes place, i.e. the increment is $\gamma = kV_0(t)/2$, then instead of (4.13) we get (at $2\beta t \ll 1$):

$$|\zeta(t)| \simeq \zeta_0 \,\exp\left\{\beta\omega(0)kR_0t^2\right\}, \tag{4.16}$$

i.e. the use of the "adiabatic expressions" (4.11) is possible at early stages of a vortex instability evolution. Similarly, in the case of "explosive" instability, with the account of (4.6) and (4.12) under the condition of $\omega(0)t \ll 1$, we obtain

$$|\zeta(t)| \simeq \zeta_0 \,\exp\left\{k\omega^2(0)R_0t^2\right\}. \tag{4.17}$$

4.3. *Development of the turbulence on the border of the tornado core*

As is known, the maximum value of the growth increment of unstable perturbations in a some non-homogeneous layer of thickness l corresponds to the wave number $k_m \simeq 1/l$. In the case of surface waves on a tangential jump of velocity, the role of the thickness of a transition non-homogeneous layer is played by the doubled amplitude of these waves (surface displacement), i.e. $k_m^{(t)} \simeq 1/2|\zeta(t)|$. Substituting this value in expression (4.16), we get the equation

$$\ln\frac{|\zeta(t)|}{\zeta_0} \simeq \frac{\beta\omega(0)R_0t^2}{2|\zeta(t)|}, \tag{4.18}$$

from which it follows that $|\zeta(t)|$ changes approximately according to the law $|\zeta(t)| \sim t^2/\ln t$. At this, the effective coefficient of the anomalous turbulent viscosity inside a surface layer of thickness $2|\zeta(t)|$ can be estimated as

$$\nu^*(t) \simeq \zeta^2(t)\gamma(t) \simeq \frac{1}{2}\beta\omega(0)R_0|\zeta(t)|t, \tag{4.19}$$

i.e. the value $\nu^*(t)$ grows in time, according to (4.18), almost by the cubic law ($\sim t^3$) and can reach very large values ($\nu^* \gg \nu$).

The saturation of turbulent perturbations occurs due to the nonlinear dissipation, when the increase of the kinetic energy of surface waves, as a result of instability of the tangential jump of velocity $\frac{1}{2}\gamma\rho\tilde{v}^2$, is compensated by the turbulent dissipation energy per unit time and unit volume $\rho\tilde{v}^3/l$ (see Landau & Lifshits (1987)), where $\tilde{v} \simeq d|\zeta(t)|/dt$ is the velocity of turbulent pulsations, $l \simeq |\zeta(t)|$ is a characteristic scale of turbulence, and $\gamma \simeq \pi V_0/|\zeta|$ is the maximum increment of the instability. Therefore, it follows that

$$\tilde{v}(t) \simeq \frac{\pi}{4} V_0(t). \tag{4.20}$$

On the other hand, the saturation of acceleration of the angular velocity of a "rigid-body" rotation of the vortex core (tornado) occurs when the azimuthal velocity $v_\varphi(t) = R_0\omega(t)$ and the value of a velocity jump $V_0(t)$ reach velocities close to the speed of sound in the air $c_s \simeq 330$ m/s, i.e. when the compressibility effects of the air and the effects of a finite bulk viscosity appear. The characteristic time of a rotation saturation of the vortex, according to (4.5), equals

$$t_{\max} \simeq \frac{1}{2\beta} \ln\left[\frac{c_s}{\omega(0)R_0}\right]. \tag{4.21}$$

Thus, $\tilde{v}_{\max} \simeq c_s$ and $\gamma_{\max} \simeq \pi c_s/\zeta_{\max}$, where the value of $\zeta_{\max}$, according to (4.18), is determined by the equation:

$$\ln\frac{\zeta_{\max}}{\zeta_0} \simeq \frac{t_{\max}^2 \beta\omega(0)R_0}{2\zeta_{\max}}. \tag{4.22}$$

Let us notice that in the case of the "explosive" regime of acceleration of a vortex, it is necessary to replace β by $\omega(0)$ in expressions (4.18), (4.19) and (4.22), and the value $t_{\max}$ under the condition $c_s \gg \omega(0)R_0$ equals $t_{\max} \simeq 1/\omega(0)$.

If we assume that the initial azimuthal and radial velocities of air on the borders of a cloud, which has radius $R_0 \simeq 1$ km, are equal by order to $v_0 = \omega(0)R_0 = \beta R_0 \simeq 10$ m/s, i.e. $\beta \simeq 10^{-2}$ s^{-1}, then the time of acceleration of the tornado core to the speed of sound c_s equals $t_{\max} \simeq (1 \div 2) \cdot 10^2$ s for the "explosive" or exponential acceleration laws of a tornado, respectively. Setting the minimum initial value of displacement ζ_0 of a surface $r = R_0$ to the free path of molecules in the air at the normal pressure $l_0 \simeq 5 \cdot 10^{-7}$ cm, we get from (4.22) the following estimation for the maximum scale of the turbulent pulsations: $\zeta_{\max} \simeq (20 \div 80)$ m.

In this case the maximum value of the turbulent viscosity in the surface layer of a thickness $2\zeta_{\max}$ on the border of a vortex core $r = R_0$ equals

$$\nu^*_{\max} \simeq \frac{1}{2}\omega(0)R_0\zeta_{\max}t_{\max}\beta \tag{4.23}$$

and can by many orders exceed the usual coefficient of the kinematic viscosity of the air. In particular, for the above mentioned estimations $t_{\max}$ and $\zeta_{\max}$ we get the value $\nu^*_{\max} \simeq 5 \cdot 10^6$ cm^2/s, whereas the usual kinematic viscosity of the air equals $\nu \simeq 0.15$ cm^2/s, i.e. their ratio is $\nu^*_{\max}/\nu \simeq 3 \cdot 10^7$.

Thus, on the surface of the central part of a powerful tornado of a radius R_0, which rotates almost with velocity of sound $V_{\varphi\max} = \omega(0)R_0 \exp[\gamma_{\max}t_{\max}] \simeq c_s$, a very viscous layer of a thickness $2\zeta_{\max} \ll R_0$ and with the viscosity $\nu^*_{\max} \gg \nu$ appears because of the turbulence. Due to this fact, a partial entrainment of the nearby air in the region $r > R_0$ occurs, and therefore the profile of azimuthal velocity distribution of the air in this region changes. We will seek it in the form

$$v_\varphi(r,t) = \Omega(r,t)R_0^2/r \quad (r > R), \tag{4.24}$$

where $\Omega(r,t)$ is an unknown function of r and t, which, according to (2.2), satisfies the equation:

$$\frac{\partial\Omega}{\partial t}-\frac{\beta R_0^2}{r}\frac{\partial\Omega}{\partial r}=\nu^*_{\max}\left(\frac{\partial^2\Omega}{\partial r^2}-\frac{1}{r}\frac{\partial\Omega}{\partial r}\right) \tag{4.25}$$

with a boundary condition $\Omega(R_0,t)=\omega(t)=\omega(0)\,e^{2\beta t}$. From here we get the following expression for the rotation velocity of the air in the external region $r>R_0$:

$$v_\varphi(r,t)=\omega(0)\,R_0\,e^{2\beta t}\left(\frac{R_0}{r}\right)^{\sigma+1}\frac{K_\sigma\left(r/l^*\right)}{K_\sigma\left(R_0/l^*\right)}, \tag{4.26}$$

where $K_\sigma(x)$ is the McDonald function (the modified Bessel function), $\sigma=(R_0/2l^*)^2-1$, and $l^*=\sqrt{\nu^*_{\max}/2\beta}$. Using the obtained above value of $\nu^*_{\max}$, we get the following estimation: $l^*\simeq 150$ m.

On large distances $r\gg l^*$ the velocity (4.26) changes according to the law

$$v_\varphi(r)\sim\left(\frac{R_0}{r}\right)^{\sigma+1}\sqrt{\frac{l^*}{r}}\,e^{-r/l^*}, \tag{4.27}$$

that explains rather a weak motion of the air around a tornado on the distances of $r>150$ m. On the other hand, those fact that the turbulent pulsations and rotation velocity of a vortex core can reach the velocity of sound ($\tilde v_{\max}\simeq V_{\varphi\max}\simeq c_s$) explains the reason of the intensive generation of low-frequency sound waves in powerful tornadoes (a "roar" of a tornado).

4.4. *Formation of a tornado funnel with account of gravity and vertical flows of the air*

As was mentioned above, for the flows of the air, which flow into a rain cloud of a cylindrical form during the time of its condensation, the following hydrodynamical velocities are natural:

$$v_r=\begin{cases}-\beta r, & r\leqslant R_0,\\ -\beta R_0^2/r, & r>R_0,\end{cases}\qquad v_\varphi=\begin{cases}\omega r, & r\leqslant R_0,\\ \omega R_0^2/r, & r>R_0,\end{cases}\qquad v_z=\begin{cases}v_{z0}+\alpha z, & r\leqslant R_0,\\ 0, & r>R_0,\end{cases} \tag{4.28}$$

where the parameter β is determined by relation (4.2). In the case of exponential instability we have $\omega(t)=\omega(0)\,e^{2\beta t}$ (for $\alpha=$ const and $|Q|=$ const). At this, bulk viscous forces in equations (2.1)–(2.3) are equal identically to zero, so that we get

$$\frac{\partial P}{\partial r}=\begin{cases}\rho r\left[\omega^2(t)-\beta^2\right], & r\leqslant R_0,\\ \dfrac{\rho R_0^4}{r^3}\left[\omega^2(0)+\beta^2\right], & r>R_0,\end{cases} \tag{4.29}$$

$$\frac{\partial P}{\partial z}=\begin{cases}-\rho\tilde g-\rho\alpha^2 z, \quad \tilde g=g+\alpha v_{z0}, & r\leqslant R_0,\\ -\rho g, & r>R_0.\end{cases} \tag{4.30}$$

Integration of equations (4.29) and (4.30) determines the difference of the pressure between two arbitrary points:

$$P_2-P_1=\begin{cases}-\rho\tilde g(z_2-z_1)-\dfrac{\rho}{2}\,\alpha^2(z_2^2-z_1^2)+\dfrac{\rho}{2}\left[\omega^2(t)-\beta^2\right](r_2^2-r_1^2), & r_{1,2}\leqslant R_0,\\ -\rho g(z_2-z_1)-\dfrac{\rho R_0^4}{2}\left[\omega^2(0)+\beta^2\right]\left(\dfrac{1}{r_2^2}-\dfrac{1}{r_1^2}\right), & r_{1,2}>R_0.\end{cases} \tag{4.31}$$

From (4.31) it follows, that the form of a surface of constant pressure (an isobar), which corresponds to the point of water drops evaporation $P_{\rm evp}$ in the internal region $r\leqslant R_0$

is determined by the equation

$$z^2(r,t) + \frac{2\tilde{g}}{\alpha^2} z(r,t) + \frac{R_0^2}{\alpha^2}\left[\omega^2(t) + \omega^2(0)\right] - \frac{r^2}{\alpha^2}\left[\omega^2(t) - \beta^2\right] - \frac{2\left(P_\infty - P_{\mathrm{evp}}\right)}{\rho\alpha^2} = 0, \tag{4.32}$$

where the coordinate z is counted from the initial flat surface $P_0 = P_\infty = P_{\mathrm{evp}}$, and in the external region $r > R_0$ is determined by the relation

$$z(r,t) = z_0(t) - \frac{\left[\omega^2(0) + \beta^2\right] R_0^4}{2gr^2} + \frac{(P_\infty - P_{\mathrm{evp}})}{\rho g}, \tag{4.33}$$

where the function $z_0(t)$ is to be obtained from the condition of the isobar continuity at the point $r = R_0$. According to equation (4.32), we find the value of the coordinate $z(r,t)$ at the point $r = R_0$:

$$z(R_0,t) = -\frac{\tilde{g}}{\alpha^2} + \sqrt{\frac{\tilde{g}^2}{\alpha^4} - \frac{\left[\omega^2(0) + \beta^2\right] R_0^2}{\alpha^2} + \frac{2\left(P_\infty - P_{\mathrm{evp}}\right)}{\rho\alpha^2}} = \mathrm{const}, \tag{4.34}$$

so that

$$z_0 = z(R_0) + \frac{R_0^2}{2g}\left[\omega^2(0) + \beta^2\right] - \frac{(P_\infty - P_{\mathrm{evp}})}{\rho g} = \mathrm{const}, \tag{4.35}$$

i.e. in the external region $r > R_0$, the form of the isobar does not depend on time.

The coordinate of a point of the isobar on the vortex axis $r = 0$, according to (4.32)–(4.35), is determined by the expression

$$z(0,t) = -\frac{\tilde{g}}{\alpha^2} + \sqrt{\frac{\tilde{g}^2}{\alpha^4} - \frac{R_0^2\left[\omega^2(t) + \omega^2(0)\right]}{\alpha^2} + \frac{2\left(P_\infty - P_{\mathrm{evp}}\right)}{\rho\alpha^2}}. \tag{4.36}$$

From the last formula it follows, that for $\tilde{g} > 0$, with the growth of the angular velocity $\omega(t)$ as a result of the exponential instability, there is an increase of the absolute value of the negative coordinate $z(0,t)$, which corresponds to the deepening of the minimum of the function $z(r,t)$ in the region $z < 0$.

In Fig. 1. cylindrically-symmetric surfaces $\tilde{z}(r,t) = z(r,t) - z(\infty,0)$ and their sections by mutually perpendicular planes depending on r for the isobar $P = P_{\mathrm{evp}}$, which corresponds to the bottom edge of a condensation area of the moisture at $r \to \infty$, are shown for 5 consecutive moments of time with the identical intervals. They display the time dynamics of a funnel evolution, which is filled with a fog (a heterogeneous mixture "air — water drops"), and develops during the origination of a tornado on the bottom edge of a cloud. The deepening of the funnel occurs till the contact of its minimum that corresponds to the value $z_{\min} = -\tilde{g}/\alpha^2$ with the surface of the Earth, or till that moment of time, when the maximum velocity of a vortex rotation reaches the speed of sound.

Thus, the account for the gravity and vertical flows allows to describe one of the main observable phenomenon — a funnel formation on the bottom edge of a cloud during the origination and development of tornados.

4.5. *Discussion of viability of the obtained results for the description of tornados and typhoons*

With the purpose of establishing of the viability of the obtained new non-stationary solutions of the Navier-Stokes and continuity equations for an incompressible viscous medium with a bulk sink and free inflow of the substance for the description of atmospheric vortices (tornados and typhoons), we will carry out numerical estimations of the

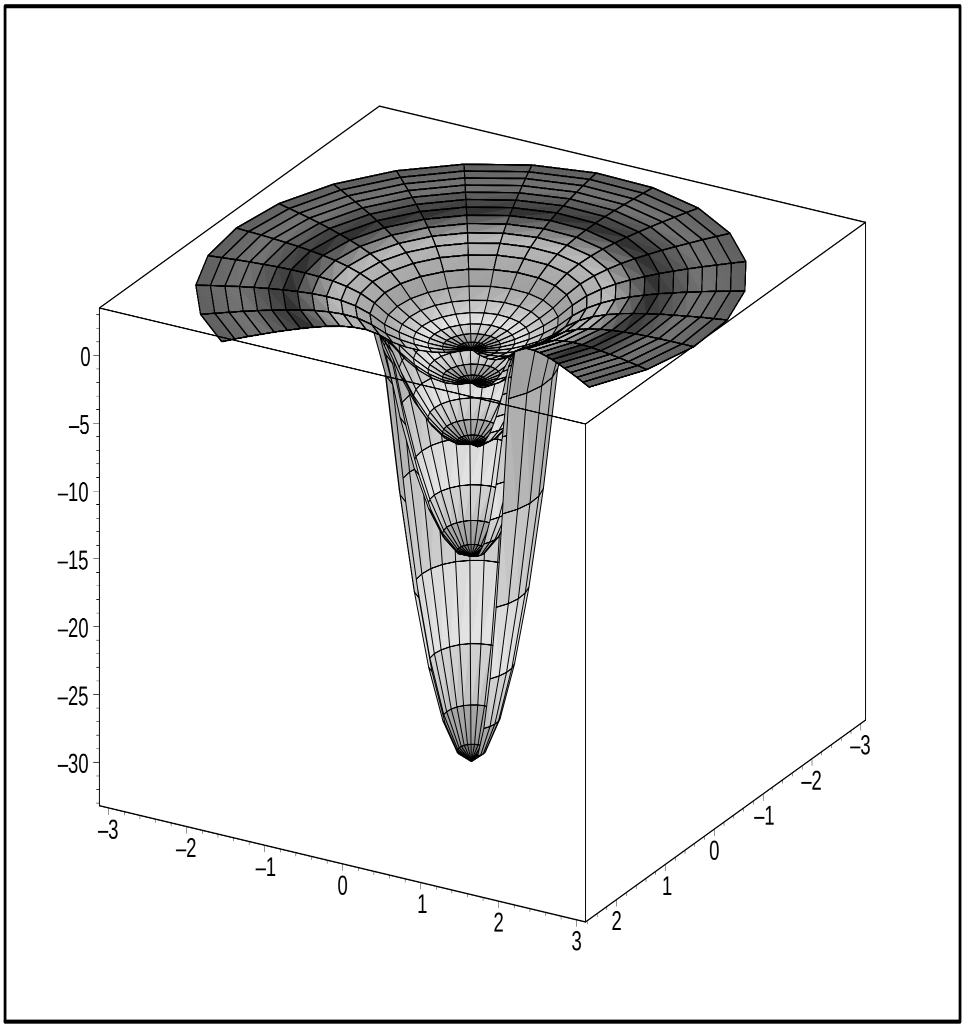

FIGURE 1. Dynamics of the isobar shape on the bottom edge of the condensation area during formation of a cloud and development of the hydrodynamical instability of a vortex (tornado). Different (descending) shapes correspond to different equidistant times of vortex evolution.

characteristic times of development of hydrodynamical instabilities of vortical motion under conditions of intensive condensation of moisture inside a cloud.

Let us consider a round cylindrical cloud of a radius $R_0 \approx 1 \div 10$ km slowly rotating in the twirled air flow with equal by order initial values of azimuthal and radial velocities on the border of a cloud $v_{r0} \approx v_{\varphi 0} \approx (1 \div 10)$ m/s. This corresponds to the absolute values of initial angular velocity $\omega(0)$ and velocities of a converging radial flow β in the range $(10^{-4} \div 10^{-2})$ s^{-1}. For humidity of atmospheric air about 100%, its density almost twice exceeds the density of the dry air $\rho_0 = 1.3 \times 10^{-3}$ g/cm^3, so that the mentioned values of the parameter β are equivalent to capacities of a bulk sink Q due to condensation of a moisture of the order of $\sim 5 \times (10^{-7} \div 10^{-5})$ g/s·cm^3. Under these conditions, the characteristic time of acceleration of a vortex rotation in the case of the exponential instability equals $\tau = 1/2\beta \approx (1 \div 100)$ min.

The maximum velocity of the order of c_s is reached in a time interval of $t \approx (3 \div 500)$

min. Typical observable times of origination and development of a tornado indeed lie in this time interval (from several minutes to several hours).

For large-scale atmospheric vortices such as cyclones, hurricanes and typhoons, which originate in cloud masses with sizes of about 100 km and more, characteristic times of the instability development increase in two – three orders of magnitude and can last for several days, which also agrees with the observable times of hurricane and typhoon existence. At the same time, the decreasing pressure on the axis of a vortex caused by the cyclostrophic rotation regime explains the characteristic "sucking" effect of a tornado.

Moreover, as marked above, in a two-phase system "air — water drops" the decrease of the pressure below the boiling point of water at the given temperature should result in the termination of the process of moisture condensation and lead to evaporation of the droplets. This explains the formation of a clear tornado "core" or a typhoon "eye" in the central part of a cloud system in the paraxial area of a vortex. Inside this area the velocity of a vortical rotation of the air slows down in time, since at the condition of $P < P_{\rm evp}$ there is a bulk source of the gas phase ($Q > 0$) instead of a bulk sink ($Q < 0$), and the parameter β changes its sign ($\beta < 0$), that corresponds to the exponential deceleration of the air, according to (3.8). Such a condition of a dead calm is observed in the center of the typhoon "eye".

It is also possible to estimate the moment of time t_H when a tornado funnel contacts with the surface of the Earth. In the case of the exponential instability, at $z = -H$ (where H is the height of the bottom edge of a cloud above the ground), according to equations (4.36) and (3.8), under the conditions $\omega^2(t) \gg \beta^2$ and $\tilde{g}/\alpha^2 \gg H$ we find

$$t_H \approx \frac{1}{4\beta} \ln \left[\frac{\tilde{g} H}{\omega^2(0) R_0^2} \right]. \tag{4.37}$$

At $H = 1$ km for the above mentioned values of $v_{\varphi 0}$ we get the estimation of $t_H \approx (2 \div 400)$ minutes.

In conclusion of this section it is necessary to emphasize, that the growth of the kinetic energy and angular momentum of a vortex during the development of the exponential or "explosive" instability does not contradict to the conservation laws of energy and angular momentum, because in an open system with a bulk sink and continuous inflow of the substance, the transfer of the necessary amount of the kinetic energy and angular momentum to the region $r \leqslant R_{\max}$ from the environment with a nonzero initial rotation velocity $v_{\varphi 0} = \omega(0) R_0^2 / r$ takes place. The value of the maximum radius $R_{\max}$ in natural conditions at tornado and typhoon origination on a free space of the land or sea is actually limited only by a finite curvature of the Earth's surface. This means that the maximum magnitudes of energy and momentum of the atmospheric vortex cannot exceed the following values (per unit length):

$$E_{\rm kin} = \pi\rho \int_{R_0}^{R_{\max}} r dr \left[v_\varphi^2(r) + v_r^2(r) \right] = \pi\rho R_0^4 \left[\omega^2(0) + \beta^2 \right] \ln\left(R_{\max}/R_0 \right), \tag{4.38}$$

$$M_z = 2\pi\rho \int_{R_0}^{R_{\max}} r^2 v_\varphi dr = \pi\rho R_0^2 R_{\max}^2 \omega^2(0). \tag{4.39}$$

For the above mentioned values of the parameters typical for tornados and for $R_{\max} = 10^3$ km we get $E_{\max} \approx (10^8 \div 10^{12})$ J/m. At this, the maximum intensity liberated during the acceleration of a vortex $\left(\dfrac{dE_{\rm kin}}{dt} \right)_{\max} \sim 4\beta E_{\max}$ reaches roughly 1 GW per meter of the vortex length, which explains the enormous destructive force of a tornado.

Thus, the carried out numerical estimations show that the introduced in the paper a new mechanism of hydrodynamical instability of a vortex in an incompressible viscous medium under the action of convective and Coriolis forces created by converging radial flows in open thermodynamically non-equilibrium systems with a bulk sink and unlimited inflow of a substance from the environment may be the real cause of origination and development of powerful atmospheric vortices — tornados and typhoons — during the intensive condensation of water vapor from the humid air cooled below the dew-point during formation of dense clouds.

5. Conclusions

In the present paper a new class of exact solutions of hydrodynamic equations for an incompressible liquid (gas) at the presence of a bulk sink and ascending flows of a substance was considered. For existence of such solutions it is essential that one or several components of a multicomponent fluid are excluded from the collective hydrodynamic motion of the liquid at the expense of chemical or phase transitions, but due to dynamical and chemical equilibrium in the open system with the surrounding medium, a constant in time and almost homogeneous in space chemical composition of the matter as well as constant density $\rho = \text{const}$ are maintained. It is shown that those profiles, which nullify the terms in the Navier-Stokes equations that describe viscous effects, exist and represent vortex structures with "rigid-body" rotation of the core and converging radial flows. In the case of constant bulk sink and inflow of the matter from the outside, the azimuthal velocity of a "rigid-body" rotation v_φ increases exponentially in time. At simultaneous infinite increase of the sink and inflow rates, v_φ increases by a scenario of the "explosive" instability, where during a finite time interval an infinite value of rotation velocity is reached.

On the basis of the developed theory of unstable hydrodynamical vortices, the offered in Pashitskii (2002) mechanism of origination and development of powerful atmospheric vortices — tornados and typhoons — during the intensive condensation of water vapor from the cooled below a dew point humid air during the formation of dense rain clouds is considered. Within the framework of this mechanism it is possible to explain the basic characteristics of tornados and typhoons. The characteristic times of development of the instability of a vortical motion agree in order of magnitude with the corresponding times of origination and existence of tornados (from several minutes to several hours) and typhoons (several days). The acceleration of a vortex rotation is limited by velocities comparable with the speed of sound, when effects of the compressibility of air, dissipative forces of a bulk viscosity and large-scale turbulence start playing their roles.

With the account of gravity, the proposed model describes the main phenomenon of a tornado — the formation of a lengthening funnel on the bottom edge of a cloud as a result of the change of the shape of the surface of a constant pressure (isobar) that bounds from the below the area of intensive condensation of moisture. It is worth noting, that in an open system there are no problems with the conservation laws of the energy and angular momentum, because the radial flows transfer not only the necessary amount of a substance, but also the necessary amount of the kinetic energy and angular momentum from the slowly rotating environment.

The velocity jump on the border of a tornado core results in the development of a strong turbulence, which can be described with the help of the anomalous coefficient of turbulent viscosity, which in many orders exceeds the usual viscosity of the air. Our estimations of the turbulent viscosity coefficient are agreed with the values known from the literature (see e.g. Kochin, Kibel & Rose (1964)).

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